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Marginal distribution of an arbitrary square submatrix of the S-matrix for Dyson's measure†

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Abstract. In a previous paper, the marginal distribution of one and two-dimensional submatrices of the S-matrix was obtained, assuming Dyson's measure for the full n-dimensional S (Dyson's measure is the invariant volume element for unitary and symmetric matrices). In the present paper we generalise the previous results and obtain the distribution of a square submatrix of arbitrary dimensionality $m \leq n/2$.

1. Introduction

In a previous paper (Pereyra and Mello 1983) the marginal distribution of individual S-matrix elements or groups of them was discussed, assuming Dyson's measure (Dyson 1962) for the full S-matrix. In particular, the distribution was given for the matrix element S_{11} and the group of four matrix elements contained in a 2×2 block along the diagonal, i.e. $\left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right)$. The procedure used there was sufficiently cumbersome that the treatment of a block with dimension greater than two was prohibitive.

In the present paper we present a different procedure which does permit the *explicit* evaluation of the joint marginal distribution of the matrix elements contained in a square submatrix (lying along the diagonal) with *arbitrary* dimensionality m (as long as $m \leq n/2$, as we shall see).

As was mentioned by Pereyra and Mello (1983), the motivation for this problem was a series of recent attempts to develop a statistical theory of nuclear reactions doing statistics directly on the S-matrix, i.e., by proposing a measure in the space of unitary and symmetric matrices (Mello 1979, Mello and Seligman 1980, Mello *et al* 1984). A very important concept in those attempts is that of Dyson's measures $d\mu(S)$, which is the invariant volume element for unitary and symmetric matrices (Dyson 1962, Hua 1963). The notion of invariance is with respect to the automorphism (U_0, unitary)

$$S \to \tilde{S} = U_0^{\mathrm{T}} S U_0, \tag{1.1}$$

that maps the space of unitary and symmetric matrices into itself. Intuitively, we can say that Dyson's measure assigns equal *a priori* probability to all unitary and symmetric matrices S of a given order n.

To be more specific about the problem to be solved in the present paper, let us call s the $m \times m$ submatrix of the n-dimensional S, the latter being distributed according to $d\mu(S)$.

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It is well known (Dyson 1962, Hua 1963) that we can write any unitary and symmetric matrix S as

$$S = U^{\mathrm{T}} U, \tag{1.2}$$

where U is unitary. If U is distributed according to the invariant or Haar measure (Hua 1962) of the unitary group $U(n)$, it follows that S is distributed according to $\mathrm{d}\mu(S)$.

Similarly, we can write the $m \times m$ submatrix s as

$$s = u^{\mathrm{T}} u, \tag{1.3}$$

where u is an $n \times m$ matrix, whose columns are the first m columns of U. Let us write these columns as m orthonormal vectors, expressed in real and imaginary parts as

$$\boldsymbol{u}_a = \boldsymbol{x}_a + \mathrm{i}\boldsymbol{y}_a, \qquad a = 1, \ldots, m. \tag{1.4}$$

This is schematically indicated in figure 1.

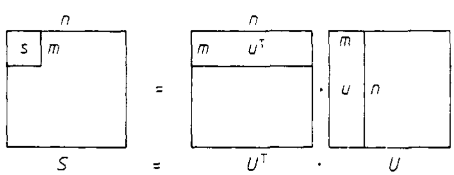

Figure 1. Schematic representation of the S-matrix, indicating the $m \times n$ matrix u, which provides the $m \times m$ submatrix s.

It is also convenient to consider explicitly the real and imaginary parts of the elements of s (which are also a part of S), defining them as X_{ij} and Y_{ij}, i.e.,

$$S_{ij} = X_{ij} + \mathrm{i}\, Y_{ij}. \tag{1.5}$$

We now define the joint differential probability of the set of variables $\{X_{ij}, Y_{ij}\}$ as

$$p_0\begin{pmatrix} X_{11}, Y_{11}, \ldots, X_{1m}, Y_{1m} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_{m1}, Y_{m1}, \ldots, X_{mm}, Y_{mm} \end{pmatrix} \cdot \prod_{a \leqslant b} (\mathrm{d}X_{ab}\, \mathrm{d}Y_{ab}), \tag{1.6}$$

where p_0 (written below as $p_0(s)$) is the probability density, i.e., the probability of finding $X_{11}, \ldots, Y_{mm}$ in a unit interval about $X_{11}, \ldots, Y_{mm}$. The index 0 is a reminder that the total S is distributed according to the invariant volume element.

Since the only constraints that Haar's measure imposes on the vectors $\boldsymbol{u}_a$ is that they be orthonormal, we can symbolically write $p_0(s)$ as

$$p_0(s) = \delta(s - s^{\mathrm{T}}) \int \delta(s - u^{\mathrm{T}} u)\, \delta(I - u^{\dagger} u)\, \mathrm{d}u. \tag{1.7a}$$

In (1.7a) we have used relation (1.3) between s and u as well as the symmetry of s.

More explicitly, by using equations (1.6) and (1.5) we write

$$
\begin{aligned}
p_0\begin{pmatrix} X_{11}, Y_{11}, \ldots, X_{1m}, Y_{1m} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ X_{m1}, Y_{m1}, \ldots, X_{mm}, Y_{mm} \end{pmatrix}
&\propto \prod_{a<b} \delta(X_{ab}-X_{ba})\,\delta(Y_{ab}-Y_{ba}) \\
&\quad\times \int \mathrm{d}^{(n)}\boldsymbol{x}_1 \ldots \int \mathrm{d}^{(n)}\boldsymbol{x}_m \int \mathrm{d}^{(n)}\boldsymbol{y}_1 \ldots \int \mathrm{d}^{(n)}\boldsymbol{y}_m \\
&\quad\times\Bigg(\prod_{a=1}^{m} \delta(X_{aa}-x_a^2+y_a^2)\,\delta(Y_{aa}-2\boldsymbol{x}_a\cdot\boldsymbol{y}_a) \\
&\quad\times \prod_{1=a<b}^{m} \delta(X_{ab}-\boldsymbol{x}_a\cdot\boldsymbol{x}_b+\boldsymbol{y}_a\cdot\boldsymbol{y}_b)\,\delta(Y_{ab}-\boldsymbol{x}_a\cdot\boldsymbol{y}_b-\boldsymbol{y}_a\cdot\boldsymbol{x}_b)\Bigg) \\
&\quad\times\Bigg(\prod_{a=1}^{m} \delta(1-x_a^2-y_a^2) \prod_{1=a<b}^{m} \delta(\boldsymbol{x}_a\cdot\boldsymbol{x}_b+\boldsymbol{y}_a\cdot\boldsymbol{y}_b)\,\delta(\boldsymbol{x}_a\cdot\boldsymbol{y}_b-\boldsymbol{y}_a\cdot\boldsymbol{x}_b)\Bigg),
\end{aligned}
\tag{1.7b}
$$

where the dots indicate scalar products between n-dimensional vectors and

$$x_a^2 \equiv \boldsymbol{x}_a\cdot\boldsymbol{x}_a, \qquad y_a^2 \equiv \boldsymbol{y}_a\cdot\boldsymbol{y}_a.$$

We prove in appendix 1 that, if $m \leq n/2$, there are enough integrations in (1.7) to eliminate all of the δ-functions (except, of course, for $\delta(s-s^{\mathrm{T}})$). We shall assume this limitation on m in what follows.

Our main goal in this paper is the exact evaluation of the expression (1.7). We shall show in the following sections that the result takes the particularly simple form

$$p_0(s) \propto \delta(s-s^{\mathrm{T}})[\det(I-s^{\dagger}s)]^{(n-2m-1)/2}. \tag{1.8}$$

For the special cases of $m = 1, 2$, equation (1.8) reduces to the results found by Pereyra and Mello (1983).

Section 2 provides a proof that $p_0(s)$ is invariant under the transformation $s = v_0^{\mathrm{T}}\tilde{s}v_0$, where v_0 is an $m \times m$ unitary matrix.

Section 3 uses this invariance property to provide an explicit evaluation of the distribution $p_0(s)$ for arbitrary $m \leq n/2$. Finally, § 4 considers the form of $p_0(s)$ in the limit of $n \gg m$ and in this limit uses the distribution for the evaluation of the average value of $|S_{ab}|^2$.

2. Invariance of $p_0(s)$

In this section we prove that the $p_0(s)$ of (1.7) remains invariant if s is replaced by $\tilde{s}$, related to s through the transformation

$$s = v_0^{\mathrm{T}}\tilde{s}v_0, \tag{2.1}$$

where v_0 is an $m \times m$ unitary matrix.

Writing

$$s = u^{\mathrm{T}}u, \qquad \tilde{s} = \tilde{u}^{\mathrm{T}}\tilde{u}, \tag{2.2}$$

as in equation (1.3), the transformation (2.1) is equivalent to

$$u = \tilde{u} v_0. \tag{2.3}$$

We recall that $u = x + \mathrm{i}y$ consists of m orthonormal vectors

$$\boldsymbol{u}_a = \boldsymbol{x}_a + \mathrm{i}\boldsymbol{y}_a \tag{2.4a}$$

and similarly

$$\tilde{\boldsymbol{u}}_a = \tilde{\boldsymbol{x}}_a + \mathrm{i}\tilde{\boldsymbol{y}}_a. \tag{2.4b}$$

First of all, from appendix 2, example 1, it follows that the volume element appearing in (1.7b) is invariant under the transformation (2.3).

Secondly, we analyse the δ-functions appearing in the curly bracket of (1.7b), indicated by $\delta(s - u^{\mathrm{T}}u)$ in (1.7a). The argument of the δ-function transforms as

$$s - u^{\mathrm{T}}u = v_0^{\mathrm{T}}(\tilde{s} - \tilde{u}^{\mathrm{T}}\tilde{u})v_0. \tag{2.5}$$

Defining the matrix on the LHS as ω and the similar matrix on the RHS as $\tilde{\omega}$, we have

$$\omega = v_0^{\mathrm{T}}\tilde{\omega}v_0. \tag{2.6}$$

Since (2.6) is linear and homogeneous we have the following relation between δ-functions of ω and of the transformed $\tilde{\omega}$:

$$\prod_{a\leqslant b} \delta(\mathrm{Re}\ \omega_{ab})\ \delta(\mathrm{Im}\ \omega_{ab})\left(\prod_{a\leqslant b} \mathrm{d}(\mathrm{Re}\ \omega_{ab})\ \mathrm{d}(\mathrm{Im}\ \omega_{ab})\right)$$
$$= \prod_{a\leqslant b} \delta(\mathrm{Re}\ \tilde{\omega}_{ab})\ \delta(\mathrm{Im}\ \tilde{\omega}_{ab})\left(\prod_{a\leqslant b} \mathrm{d}(\mathrm{Re}\ \tilde{\omega}_{ab})\ \mathrm{d}(\mathrm{Im}\ \tilde{\omega}_{ab})\right). \tag{2.7}$$

Applying the result of appendix 2, example 2, we find that the volume elements appearing in (2.7) are equal, from which it follows that the products of the δ-functions are equal. In the notation of (1.7a) this means that we can symbolically write

$$\delta(\omega) = \delta(\tilde{\omega}), \tag{2.7a}$$

or

$$\delta(s - u^{\mathrm{T}}u) = \delta(\tilde{s} - \tilde{u}^{\mathrm{T}}\tilde{u}). \tag{2.7b}$$

By a similar argument we prove the invariance of the δ-functions appearing in front of the integral in (1.7b), i.e. of $\delta(s - s^{\mathrm{T}})$ in (1.7a).

Finally, we analyse the δ-functions appearing in the square bracket of (1.7b), indicated as $\delta(I - u^{\dagger}u)$ in (1.7a). Defining the hermitian matrix $h = I - u^{\dagger}u$, (2.3) implies the transformation

$$h = v_0^{\dagger}\tilde{h}v_0. \tag{2.8}$$

Since (2.8) is linear and homogeneous we have the relation between δ-functions, similar to equation (2.7),

$$\prod_{a\leqslant b} \delta(\mathrm{Re}\ h_{ab})\ \delta(\mathrm{Im}\ h_{ab})\left(\prod_{a\leqslant b} \mathrm{d}(\mathrm{Re}\ h_{ab})\ \mathrm{d}(\mathrm{Im}\ h_{ab})\right)$$
$$= \prod_{a\leqslant b} \delta(\mathrm{Re}\ \tilde{h}_{ab})\ \delta(\mathrm{Im}\ \tilde{h}_{ab})\left(\prod_{a\leqslant b} \mathrm{d}(\mathrm{Re}\ \tilde{h}_{ab})\ \mathrm{d}(\mathrm{Im}\ \tilde{h}_{ab})\right). \tag{2.9}$$

We now apply the result of appendix 2, example 3: the volume elements appearing in (2.9) are equal, so that equating the δ-functions we can write symbolically

$$\delta(h)=\delta(\tilde{h}) \tag{2.10a}$$

or

$$\delta(I-u^{\dagger}u)=\delta(I-\tilde{u}^{\dagger}\tilde{u}). \tag{2.10b}$$

In conclusion, we can write (1.7*a*) as

$$\begin{aligned} p_0(s) &= \delta(s-s^{\mathrm{T}}) \int \delta(s-u^{\mathrm{T}}u)\,\delta(I-u^{\dagger}u)\,\mathrm{d}u \\ &= \delta(\tilde{s}-\tilde{s}^{\mathrm{T}}) \int \delta(\tilde{s}-\tilde{u}^{\mathrm{T}}\tilde{u})\,\delta(I-\tilde{u}^{+}\tilde{u})\,\mathrm{d}\tilde{u}=p_0(\tilde{s}), \end{aligned}$$

which proves the invariance of $p_0(s)$.

3. Evaluation of the distribution $p_0(s)$ of (1.7)

We have proved in § 2 that $p_0(s)$ remains invariant if s is subject to the transformation (2.1). Therefore, if we need p_0 for a certain s, we may look for a convenient v_0 that takes s into an $\tilde{s}$ where the calculation of p_0 is simpler, and then apply the invariance relation $p_0(s)=p_0(\tilde{s})$. In particular, we can always find v_0 so that s is diagonal and real (Engelbrecht and Weidenmüller 1973). As we shall see, this choice considerably simplifies the calculation of equation (1.7). We thus evaluate (1.7*b*) for the case in which only the diagonal elements X_{aa} $(a=1,\ldots,m)$ may be different from zero (because of the symmetry δ-functions on the RHS of (1.7*b*) we must still keep explicit reference to the elements S_{ab}, $a>b$). For simplicity, we shall drop the tilde from $\tilde{s}$ in the analysis that follows. The differential volume elements in (1.7*b*) can be expressed in hyperspherical coordinates, to give

$$\begin{aligned} p_0 \propto & \prod_{a>b} \delta(X_{ab})\,\delta(Y_{ab}) \\ & \times \int \prod_{a=1}^{m} [\delta(X_{aa}-x_a^2+y_a^2)\,\delta(1-x_a^2-y_a^2)] \prod_{a=1}^{m} \delta(\boldsymbol{x}_a \cdot \boldsymbol{y}_a) \\ & \times \prod_{a<b} [\delta(\boldsymbol{x}_a\cdot\boldsymbol{x}_b-\boldsymbol{y}_a\cdot\boldsymbol{y}_b)\,\delta(\boldsymbol{x}_a\cdot\boldsymbol{y}_b+\boldsymbol{y}_a\cdot\boldsymbol{x}_b)\,\delta(\boldsymbol{x}_a\cdot\boldsymbol{x}_b+\boldsymbol{y}_a\cdot\boldsymbol{y}_b) \\ & \times \delta(\boldsymbol{x}_a\cdot\boldsymbol{y}_b-\boldsymbol{y}_a\cdot\boldsymbol{x}_b)] \prod_{a=1}^{m} (x_a^{n-1}\,\mathrm{d}x_a\,\mathrm{d}\Omega_a^{(x)}\,y_a^{n-1}\,\mathrm{d}y_a\,\mathrm{d}\Omega_a^{(y)}). \end{aligned} \tag{3.1}$$

The δ-functions in the first square bracket imply that $x_a^2=(1+X_{aa})/2$, $y_a^2=(1-X_{aa})/2$; those in the second square bracket imply that $\boldsymbol{x}_a\cdot\boldsymbol{x}_b=\boldsymbol{y}_a\cdot\boldsymbol{y}_b=\boldsymbol{x}_a\cdot\boldsymbol{y}_b=\boldsymbol{y}_a\cdot\boldsymbol{x}_b=0$. Therefore

$$\begin{aligned} p_0 \propto & \prod_{a>b} \delta(X_{ab})\,\delta(Y_{ab}) \int \prod_{a=1}^{m} \left[\delta\left(x_a^2-\frac{1+X_{aa}}{2}\right)\delta\left(y_a^2-\frac{1-X_{aa}}{2}\right)\right] \\ & \times \prod_{a=1}^{m} \delta(\boldsymbol{x}_a\cdot\boldsymbol{y}_a) \prod_{a<b} [\delta(\boldsymbol{x}_a\cdot\boldsymbol{x}_b)\,\delta(\boldsymbol{y}_a\cdot\boldsymbol{y}_b)\,\delta(\boldsymbol{x}_a\cdot\boldsymbol{y}_b)\,\delta(\boldsymbol{y}_a\cdot\boldsymbol{x}_b)] \\ & \times \prod_{a=1}^{m} x_a^{n-2}\,\mathrm{d}(x_a^2)\,\mathrm{d}\Omega_a^{(x)}\,y_a^{n-2}\,\mathrm{d}(y_a^2)\,\mathrm{d}\Omega_a^{(y)}. \end{aligned} \tag{3.2}$$

The integrals over the magnitudes x_a^2, y_a^2 can be done immediately, leaving only the integrals over the solid angles. With the notation $\hat{x}_a = x_a/x_a$, $\hat{y}_a = y_a/y_a$ to indicate unit vectors, we have

$$
\begin{aligned}
p_0 \propto \prod_{a>b} [\delta(X_{ab})\,\delta(Y_{ab})] \prod_{a=1}^{m} \left(\frac{1+X_{aa}}{2}\frac{1-X_{aa}}{2}\right)^{-1/2} \\
\times \prod_{a<b} \left[\left(\frac{1+X_{aa}}{2}\frac{1+X_{bb}}{2}\right)^{-1/2}\left(\frac{1-X_{aa}}{2}\frac{1-X_{bb}}{2}\right)^{-1/2}\right. \\
\left.\times\left(\frac{1+X_{aa}}{2}\frac{1-X_{bb}}{2}\right)^{-1/2}\left(\frac{1-X_{aa}}{2}\frac{1+X_{bb}}{2}\right)^{-1/2}\right] \\
\times \prod_{a}\left(\frac{1+X_{aa}}{2}\frac{1-X_{aa}}{2}\right)^{(n-2)/2} \int \prod_{a} \delta(\hat{x}_a\cdot\hat{y}_a) \\
\times \prod_{a<b} [\delta(\hat{x}_a\cdot\hat{x}_b)\,\delta(\hat{y}_a\cdot\hat{y}_b)\,\delta(\hat{x}_a\cdot\hat{y}_b)\,\delta(\hat{y}_a\cdot\hat{x}_b)] \prod_{a} [\mathrm{d}\Omega_a^{(x)}\,\mathrm{d}\Omega_a^{(y)}]. \qquad (3.3)
\end{aligned}
$$

The remaining integral over the solid angles is a constant, independent of the argument of p_0. This is the great simplification that we achieve with the special choice of s that we have made. We thus have

$$
\begin{aligned}
p_0 \propto \prod_{a>b} [\delta(X_{ab})\,\delta(Y_{ab})] \\
\times\left(\prod_{a}(1-X_{aa}^2)^{(n-3)/2} \Big/ \prod_{a<b}[(1-X_{aa}^2)(1-X_{bb}^2)]\right). \qquad (3.4)
\end{aligned}
$$

Using the identity

$$
\prod_{1=a<b}^{m} (c_a c_b) = \prod_{a=1}^{m} c_a^{m-1}, \qquad (3.5)
$$

we can then write

$$
p_0(X_{11}, X_{22}, \ldots, X_{mm}, \{S_{ab}, a>b\}) \propto \prod_{a>b} [\delta(X_{ab})\,\delta(Y_{ab})] \prod_{a=1}^{m} (1-X_{aa}^2)^{(n-2m-1)/2}. \qquad (3.6)
$$

It is clear that $\Pi_a(1-X_{aa}^2)$ can be written in a form which is invariant under the transformation (2.1) as $\det(I-s^\dagger s)$. We thus reach the final result for the distribution $p_0(s)$ quoted in equation (1.8)

$$
p_0(s) \propto \delta(s-s^{\mathrm{T}})[\det(I-s^\dagger s)]^{(n-2m-1)/2}. \qquad (3.7)
$$

4. The distribution $p_0(s)$ for $n \gg m$

We can write equation (3.7) as

$$
\begin{aligned}
p_0(s) &\propto \delta(s-s^{\mathrm{T}}) \exp[\tfrac{1}{2}(n-2m-1)\,\mathrm{Tr}\ln(I-s^\dagger s)] \\
&= \delta(s-s^{\mathrm{T}}) \exp\left(-\tfrac{1}{2}(n-2m-1)\sum_{k=1}^{\infty}\frac{1}{k}(\mathrm{Tr}\, s^\dagger s)^k\right), \qquad (4.1)
\end{aligned}
$$

where the series is convergent since s is a subunitary matrix.

In the limit $n \gg m$, expression (4.1) is dominated by the leading term, so that

$$p_0(s) \propto \delta(s - s^{\mathrm{T}}) \exp[-\tfrac{1}{2}n \operatorname{Tr}(s^{\dagger} s)]. \tag{4.2}$$

Writing each of the matrix elements of s explicitly, we thus have

$$p_0(\{S_{ij};\, i, j = 1, \ldots, m\}) \propto \prod_{a<b} \delta(S_{ab} - S_{ba}) \exp\left(-\frac{n}{2}\sum_{ab} |S_{ab}|^2\right), \tag{4.3}$$

or

$$p_0(\{S_{ij};\, i \geqslant j = 1, \ldots, m\}) \propto \exp\left[-\frac{n}{2}\left(\sum_{a=1}^{m} |S_{aa}|^2 + 2 \sum_{1=a<b}^{m} |S_{ab}|^2\right)\right]. \tag{4.4}$$

Thus, in the $n \gg m$ limit, the variables S_{ij} $(i \geqslant j = 1, \ldots, m)$ are distributed as independent zero-centred Gaussian variables.

As an example, let us use the probability density given in equation (4.4) to evaluate the average value of $|S_{ab}|^2$, i.e. $\langle |S_{ab}|^2 \rangle_0$, where the index 0 emphasises that the total S-matrix is distributed according to the invariant volume element.

We find from (4.4)

$$\langle |S_{ab}|^2 \rangle_0 \approx (1 + \delta_{ab})/n. \tag{4.5}$$

This variance (but not the entire distribution) was obtained by Mello and Seligman (1980) for arbitrary n as

$$\langle |S_{ab}|^2 \rangle_0 = (1 + \delta_{ab})/(n + 1).$$

Further applications of the marginal distribution derived here have been made by Friedman and Mello (1984).

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Appendix 1. Proof that if $m \leqslant n/2$, the δ-functions in equation (1.7*b*) can be eliminated

Inside the integral (1.7*b*) there are $m(m+1)$ δ-functions containing s-matrix elements and m^2 coming from the orthonormality of the $\boldsymbol{u}_a$'s. We thus have $2m^2 + m$ δ-functions and $2mn$ integrations, so that we obtain, at this stage, the inequality

$$2m^2 + m < 2mn, \qquad \text{or} \quad m < n - \tfrac{1}{2}, \tag{A1.1}$$

in order to eliminate the δ-functions. However, we shall show in what follows that not all of the $2mn$ integrations contribute to eliminate the δ-functions. To see this, we will find it advantageous to express the vectors $\boldsymbol{x}_a$ and $\boldsymbol{y}_a$ in spherical coordinates:

$$\boldsymbol{x}_a: \quad x_a, \theta_a^{(1)}, \ldots, \theta_a^{(n-1)}; \qquad a = 1, \ldots, m \tag{A1.2a}$$

$$\boldsymbol{y}_a: \quad y_a, \varphi_a^{(1)}, \ldots, \varphi_a^{(n-1)}; \qquad a = 1, \ldots, m \tag{A1.2b}$$

with the corresponding volume elements

$$\mathrm{d}^{(n)}\boldsymbol{x}_a = x_a^{n-1} \sin^{n-2}\theta_a^{(1)} \ldots \sin\theta_a^{(n-2)}\, \mathrm{d}x_a\, \mathrm{d}\theta_a^{(1)} \ldots \mathrm{d}\theta_a^{(n-1)} \tag{A1.3a}$$

$$\mathrm{d}^{(n)}\boldsymbol{y}_a = y_a^{n-1} \sin^{n-2}\varphi_a^{(1)} \ldots \sin\varphi_a^{(n-2)}\, \mathrm{d}y_a\, \mathrm{d}\varphi_a^{(1)} \ldots \mathrm{d}\varphi_a^{(n-1)}. \tag{A1.3b}$$

Since the integrand in (1.7b) depends only on the scalar products of the vector $\boldsymbol{x}_a, \boldsymbol{y}_b$ $(a, b = 1, \ldots, m)$, we can choose the coordinate system so that $\boldsymbol{x}_1$ has only one component different from zero, $\boldsymbol{y}_1$ only two, etc. For a given m, suppose that we choose $n \geqslant 2m$. We have:

$$\begin{array}{llllll}
\boldsymbol{x}_1 = (x_1)\ (1, & 0 & 0 & 0 & 0 & 0\ldots) \\
\boldsymbol{y}_1 = (y_1)\ (\cos\varphi_1^{(1)}, & \sin\varphi_1^{(1)}, & 0 & 0 & 0 & 0\ldots) \\
\boldsymbol{x}_2 = (x_2)\ (\cos\theta_2^{(1)}, & \sin\theta_2^{(1)}\cos\theta_2^{(2)}, & \sin\theta_2^{(1)}\sin\theta_2^{(2)}, & 0 & 0 & 0\ldots) \\
\boldsymbol{y}_2 = (y_2)\ (\cos\varphi_2^{(1)}, & \sin\varphi_2^{(1)}\cos\varphi_2^{(2)}, & \sin\varphi_2^{(1)}\sin\varphi_2^{(2)}\cos\varphi_2^{(3)}, & \sin\varphi_2^{(1)}\sin\varphi_2^{(2)}\sin\varphi_2^{(3)}, & 0 & 0\ldots) \\
\vdots & & \vdots & & \vdots & \vdots \\
\boldsymbol{x}_m = (x_m)(\cos\theta_m^{(1)}, & \sin\theta_m^{(1)}\cos\theta_m^{(2)}, & \sin\theta_m^{(1)}\sin\theta_m^{(2)}\cos\theta_m^{(3)}, \ldots, & \sin\theta_m^{(1)}\ldots\sin\theta_m^{(2m-2)}, & 0 & 0\ldots) \\
\boldsymbol{y}_m = (y_m)(\cos\varphi_m^{(1)}, & \sin\varphi_m^{(1)}\cos\varphi_m^{(2)}, & \sin\varphi_m^{(1)}\sin\varphi_m^{(2)}\cos\varphi_m^{(3)}, \ldots, & \sin\varphi_m^{(1)}\ldots\sin\varphi_m^{(2m-2)}\cos\varphi_m^{(2m-1)}, & \sin\varphi_m^{(1)}\ldots\sin\varphi_m^{(2m-1)}, & 0\ldots)
\end{array} \tag{A1.4}$$

Of all the variables appearing in (A1.2), only those indicated explicitly in (A1.4) appear in the integrand in (1.7b). In (A1.4) we have $2m$ radial variables; associated with the various unit vectors, we have $(2m-1)$ independent angular variables in the first component (i.e., $\varphi_1^{(1)}, \theta_2^{(1)}, \ldots, \theta_m^{(1)}, \varphi_m^{(1)}$), $(2m-2)$ additional independent variables in the second one, etc, zero new ones in the $2m$th component, which is the last component of $\boldsymbol{y}_m$ that is different from zero. Altogether we then have

$$2m + [(2m-1) + (2m-2) + \ldots + 1 + 0] = 2m^2 + m \tag{A1.5}$$

variables. Therefore, out of the $2mn$ integrations, only $2m^2 + m$ are 'effective' in eliminating the δ-functions; the number of δ-functions is precisely $2m^2 + m$, according to what was stated before equation (A1.1). For the same m, if we had $n = 2m - 1$, the last component of $\boldsymbol{y}_m$ in (A1.4) would not be present; φ_{2m-1} would not be necessary and the number of variables would be one less than before, whereas the number of δ-functions is still the same; in this case, one δ-function still survives. This proves our statement that n must be at least equal to $2m$ to get rid of all the δ-functions.

Appendix 2. Invariance property of the volume elements

We consider in this appendix the transformation properties of some of the volume elements used in this paper.

(1) Let z be an $n \times m$ *complex matrix*

$$z = x + \mathrm{i}y, \tag{A2.1}$$

which we subject to the transformation

$$z = \tilde{z} v_0, \tag{A2.2}$$

where v_0 is a unitary $m\times m$ matrix and

$$\tilde{z}=\tilde{x}+\mathrm{i}\tilde{y}. \tag{A2.3}$$

We represent (A2.2) as

$$\begin{array}{cc} & m \\ n & \boxed{\;z\;} \end{array} = \begin{array}{cc} & m \\ n & \boxed{\;\tilde{z}\;} \end{array} \quad \begin{array}{cc} m & \\ \boxed{v_0} & m \end{array} \tag{A2.4}$$

We now prove the invariance of the volume element

$$\prod_{b=1}^{m}\prod_{a=1}^{n} \mathrm{d}x_{ab}\,\mathrm{d}y_{ab}=\prod_{b=1}^{m}\prod_{a=1}^{n} \mathrm{d}\tilde{x}_{ab}\,\mathrm{d}\tilde{y}_{ab}. \tag{A2.5}$$

Writing v_0 in terms of its real and imaginary parts

$$v_0=\xi+\mathrm{i}\eta, \tag{A2.6}$$

(A2.2) gives

$$x=\tilde{x}\xi-\tilde{y}\eta \tag{A2.7a}$$

$$y=\tilde{y}\xi+\tilde{x}\eta, \tag{A2.7b}$$

which can be arranged in matrix form as

$$\begin{array}{ccc} & m & m \\ n & \boxed{\;x\;} & \boxed{\;y\;} \end{array} = \begin{array}{ccc} m & m & \\ \boxed{\;\tilde{x}\;} & \boxed{\;\tilde{y}\;} & n \end{array} \quad \begin{array}{ccc} & m & m \\ m & \xi & \eta \\ m & -\eta & \xi \end{array} \tag{A2.8}$$

The volume elements are then related through

$$\prod_{ab} \mathrm{d}x_{ab}\,\mathrm{d}y_{ab}=\left|\det\begin{pmatrix} \xi & \eta \\ -\eta & \xi \end{pmatrix}\right|^{n} \prod_{ab} \mathrm{d}\tilde{x}_{ab}\,\mathrm{d}\tilde{y}_{ab}. \tag{A2.9}$$

The determinant in (A2.9) can be calculated as

$$\begin{aligned} \left|\det\begin{pmatrix} \xi & \eta \\ -\eta & \xi \end{pmatrix}\right| &= \left|\det\begin{pmatrix} \xi^{\mathrm{T}} & -\eta^{\mathrm{T}} \\ \eta^{\mathrm{T}} & \xi^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} \xi & \eta \\ -\eta & \xi \end{pmatrix}\right|^{1/2} \\ &= \left|\det\begin{pmatrix} \xi^{\mathrm{T}}\xi+\eta^{\mathrm{T}}\eta & \xi^{\mathrm{T}}\eta-\eta^{\mathrm{T}}\xi \\ \eta^{\mathrm{T}}\xi-\xi^{\mathrm{T}}\eta & \eta^{\mathrm{T}}\eta+\xi^{\mathrm{T}}\xi \end{pmatrix}\right|^{1/2} \\ &= \left|\det\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right|^{1/2}=1, \end{aligned} \tag{A2.10}$$

where we have used the unitarity of v_0

$$v_0^\dagger v_0 = (\xi^T - i\eta^T)(\xi + i\eta) = (\xi^T\xi + \eta^T\eta) + i(\xi^T\eta - \eta^T\xi) = I. \tag{A2.11}$$

Insertion of (A2.10) in (A2.9) proves our statement (A2.5).

(2) Let s be an $m \times m$ *complex symmetric matrix*

$$s = X + iY, \tag{A2.12}$$

which we subject to the transformation

$$s = v_0^T \tilde{s} v_0, \tag{A2.13}$$

where v_0 is a unitary $m \times m$ matrix and

$$\tilde{s} = \tilde{X} + i\tilde{Y}. \tag{A2.14}$$

We prove the invariance of the volume element

$$\prod_{a \leqslant b} (dX_{ab}\, dY_{ab}) = \prod_{a \leqslant b} (d\tilde{X}_{ab}\, d\tilde{Y}_{ab}). \tag{A2.15}$$

Notice that since s and $\tilde{s}$ are symmetric, only the differentials of the independent matrix elements occur in (A2.15), and hence the condition $a \leqslant b$. This restriction did not occur in problem (1) of this appendix, in connection with (A2.5), so that the proof is now more difficult.

We first write (A2.13) in components

$$s_{ab} = \sum_{cd} v_{ca}^0 v_{db}^0 \tilde{s}_{cd}. \tag{A2.16}$$

Suppose that v_0 differs only infinitesimally from the unit matrix, so that

$$v_{ab}^0 = \delta_{ab} + \varepsilon_{ab}, \tag{A2.17}$$

with $\varepsilon = -\varepsilon^\dagger$. If we can prove our statement for this case, the proof for any unitary v_0 follows from exponentiation.

To first order in ε we can write (A2.16) as

$$s_{ab} = (1 + \varepsilon_{aa} + \varepsilon_{bb})\tilde{s}_{ab} + \sum_{\substack{c \leqslant d \\ (\neq a,b)}} O(\varepsilon) s_{cd}, \tag{A2.18}$$

where the coefficient of the second term is of order ε. Writing $\varepsilon_{aa} + \varepsilon_{bb} \equiv \eta_{ab} = \eta'_{ab} + i\eta''_{ab}$ we have

$$X_{ab} = (1 + \eta'_{ab})\tilde{X}_{ab} - \eta''_{ab}\tilde{Y}_{ab} + \sum_{\substack{c \leqslant d \\ (\neq a,b)}} O(\varepsilon)\tilde{X}_{cd} + \sum_{\substack{c \leqslant d \\ (\neq a,b)}} O(\varepsilon)\tilde{Y}_{cd} \tag{A2.19a}$$

$$Y_{ab} = \eta''\tilde{X}_{ab} + (1 + \eta'_{ab})\tilde{Y}_{ab} + \sum_{\substack{c \leqslant d \\ (\neq a,b)}} O(\varepsilon)\tilde{X}_{cd} + \sum_{\substack{c \leqslant d \\ (\neq a,b)}} O(\varepsilon)\tilde{Y}_{cd}. \tag{A2.19b}$$

These relations can be put in matrix form, as shown in figure 2.

$$
\begin{bmatrix} X_{ab} \\ Y_{ab} \\ X_{a'b'} \\ Y_{a'b'} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}
=
\begin{bmatrix}
\begin{array}{|c|c|}\hline 1+\eta'_{ab} & -\eta''_{ab} \\ \hline \eta''_{ab} & 1+\eta'_{ab} \\ \hline \end{array} & & & O(\varepsilon) \\
& \begin{array}{|c|c|}\hline 1+\eta'_{a'b'} & -\eta''_{a'b'} \\ \hline \eta''_{a'b'} & 1+\eta'_{a'b'} \\ \hline \end{array} & & \\
& & \ddots & \\
O(\varepsilon) & & & \cdot
\end{bmatrix}
\begin{bmatrix} \tilde{X}_{ab} \\ \tilde{Y}_{ab} \\ \tilde{X}_{a'b'} \\ \tilde{Y}_{a'b'} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}
$$

Figure 2. Schematic representation of the transformation (A2.19) relating $\{\tilde{X}_{ab}, \tilde{Y}_{ab}\}$ to $\{X_{ab}, Y_{ab}\}$.

The matrix indicated is $m(m+1)$-dimensional. The elements outside the blocks are of order ε. The Jacobian J we are looking for is the determinant of this matrix which, to first order in ε, is given by

$$
\begin{aligned}
J &= \prod_{a\leqslant b} (1+2\eta'_{ab}) \approx 1+2\sum_{a\leqslant b} \eta'_{ab} \\
&= 1+2\left(\sum_a \eta'_{aa} + \sum_{a<b} \eta'_{ab}\right) \\
&= 1+2\left(\sum_a 2\varepsilon'_{aa} + \sum_{a<b} (\varepsilon'_{aa}+\varepsilon'_{bb})\right),
\end{aligned} \tag{A2.20}
$$

where ε' denotes Re ε. Since ε is anti-hermitian, $\varepsilon'_{aa}=0$ and $J=1$. This proves our statement.

(3) Let h be an *$m\times m$ hermitian matrix*

$$h = X + \mathrm{i}\, Y, \tag{A.2.21}$$

which we subject to the transformation

$$h = v_0^{\dagger} \tilde{h} v_0, \tag{A.2.22}$$

where v_0 is a unitary matrix and

$$\tilde{h} = \tilde{X} + \mathrm{i}\, \tilde{Y}. \tag{A.2.23}$$

We prove the invariance of the volume element

$$\prod_{a\leqslant b} (\mathrm{d}X_{ab}\, \mathrm{d}Y_{ab}) = \prod_{a\leqslant b} (\mathrm{d}\tilde{X}_{ab}\, \mathrm{d}\tilde{Y}_{ab}). \tag{A2.24}$$

The proof is similar to that of example (2) above. With a v_0 differing infinitesimally from the unit matrix, as in equation (A2.17), (A2.22) gives

$$h_{ab} = (1+\varepsilon^*_{aa}+\varepsilon_{bb})\tilde{h}_{ab} + \sum_{\substack{c\leqslant d \\ (\neq a,b)}} O(\varepsilon)\tilde{h}_{cd}. \tag{A2.25}$$

Defining $\eta_{ab} \equiv \varepsilon^*_{aa} + \varepsilon_{bb} = \eta'_{ab} + \mathrm{i}\,\eta''_{ab}$, equations (A2.19) and (A2.20) hold in the present case also. Again, $\varepsilon'_{aa}=0$ and $J=1$, which proves the invariance (A2.24).

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